Section 4.6

Linear Approximation

As mentioned earlier in class, one interpretation of the tangent line to a curve is the straight line that most closely resembles the curve. This is due to the instantaneous rate of change for both the line and the curve being equal for the x value where they touch. We will explore this approximation more fully in this section.

The first key to form linear approximations is differentiability. Compare the graphs of $f(x) = x^2$ and $g(x) = |x|$. Which looks most, or at all, like a straight line near the point $(0, 0)$?

Definition 4.6.1. Suppose f is differentiable on an open interval I containing the point a. The equation of the tangent line of f at the point $(a, f(a))$ is given by $y = f'(a)(x - a) + f(a)$. The *linear approximation* to f at a is the linear function

$$y = L(x) = f'(a)(x - a) + f(a)$$

defined for x in I.

Example 4.6.2. Suppose you record the time it takes for you to drive a mile while traveling a constant speed. Write a function $s(x)$ that computes your speed for x seconds over or under a minute. Use the linear approximation of $s(x)$ to approximate your speed if it takes 62 seconds to drive a mile.

Example 4.6.3. Use linear approximation of $\sin x$ to approximate $\sin(0.0001)$.

The closer the x value is to a, the closer $L(x)$ is to $f(x)$. But there is always some error, and the cause is f not being straight. That is, it is concave up/down.

Theorem 4.6.4. Let f be differentiable on an open interval I containing the point a.

- If $f''(x) > 0$ for all x in I, then $L(x) < f(x)$ (L is an underestimate of f)
- If $f''(x) < 0$ for all x in I, then $L(x) > f(x)$ (L is an overestimate of f)

If M is the absolute maximum of f'' on I, then

$$E(x) = |L(x) - f(x)| \leq \frac{M(x-a)^2}{2}$$

Sketch.

Example 4.6.5. Find the linear approximation of $f(x) = x^{\frac{1}{3}}$ at $x = 1$ and use it to approximate $f(2)$. Is this approximation an over- or underestimate?

Measuring Change

The standard symbol to denote change is Δ. Given a function f, for a change in x-coordinate, written $\Delta x = x - a$, we have a corresponding change in y-coordinate written $\Delta y = f(x) - f(a)$.

Rewriting the linear approximation $f(x) \approx f'(a)(x - a) + f(a)$ we have

$$\Delta y \approx f'(a)\Delta x$$

Example 4.6.6. Approximate the change in surface area of a sphere when the radius decreases from 4 m to 3.9 m.

For the linear approximation $L(x)$ of f at a, the change in L, ΔL is exactly (no approximation necessary)

$$\Delta L = f'(a)\Delta x$$

In order to distinguish Δy and ΔL we write $\Delta L = dy$ which we call the *differential* and at the same time write $\Delta x = dx$.

Definition 4.6.7. Let f be differentiable on an open interval containing x. A small change in x is denoted by the differential dx. The corresponding change in f is approximately equal to the differential $dy = f'(x)dx$; that is,

$$\Delta y \approx dy = f'(x)dx.$$

Example 4.6.8. Use differentials to express the relationship between a small change in x with a small change in y for the functions

1. $f(x) = \sin^2 x$

2. $g(x) = \ln(1 - x)$